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A trihamiltonian extension of the Toda lattice

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Abstract

A new Poisson structure is defined on a subspace of the Kupershmidt algebra, isomorphic to the space $\mathfrak{H}$ of $n \times n$ Hermitian matrices. The new Poisson structure is of Lie–Poisson type with respect to the standard Lie bracket of $\mathfrak{H}$. This Poisson structure (together with two already known ones, obtained through a r-matrix technique) allows to construct an extension of the periodic Toda lattice with n particles that fits in a trihamiltonian recurrence scheme. Some explicit examples of the construction and of the first integrals found in this way are given.

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1. Introduction

The Toda lattice is a fundamental example of a completely integrable system. It consists of n particles on a line (in the non-periodic case) or on a circle (in the periodic case), each interacting with its nearest neighbours through an exponential repulsive force. Hence, the Hamiltonian of the non-periodic Toda lattice with n particles is:

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n-1} e^{q_k - q_{k+1}}.$$

The periodic case is obtained by allowing the second sum to run from $k = 1$ to n and by setting $q_{n+1} = q_1$.

This lattice was first introduced by Toda in 1967; its integrability was proved in 1974 by Flaschka, Hénon and Manakov [11,12,17]. Moser [22] found an explicit solution for the non-periodic case in 1975. In 1976 Bogoyavlensky [3] generalized the system by constructing analogous integrable lattices on any simple algebra. Kostant in [13] showed an equivalence between the integration of the Toda lattice and the representation theory of simple Lie algebras. The Toda lattice, either periodic or non-periodic, has been studied through r-matrix and

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bihamiltonian methods, becoming a standard example in the theory of completely integrable systems. Many aspects of the system were analyzed, including master symmetries, recursion operators, action-angle variables and multiple Poisson brackets [4,5,7]. Moreover many reformulations and generalizations were given: beside the already cited constructions on simple algebras, it is worth citing the relativistic Toda lattice [25], the full Kostant–Toda lattice [8], the extension of the system related to Lie algebroids [19,20] and the Kupershmidt formulation of the periodic lattice in terms of shift operators [14]. In recent years the problem of the separability of the Toda lattice has been approached with the classical technique of point transformations [18] and from the point of view of bihamiltonian theory [15,10,9].

Both the periodic and the non-periodic Toda lattices admit a trihamiltonian formulation (indeed a multi-Hamiltonian one, see [4] and references therein), in the sense that three (compatible) Poisson structures P, Q, and S exist such that, for a suitable set of Hamiltonians $\{h_k\}$, the recursion relations

$$X_k = P\mathrm{d}h_k = Q\mathrm{d}h_{k+1} = S\mathrm{d}h_{k-1} \tag{1}$$

hold. The Poisson tensors P, Q and S arise from a fairly general construction on Lie algebras, equipped with a r-matrix and an underlying associative product [23]. In particular for the periodic Toda lattice this construction is performed on the Kupershmidt algebra [21,14].

In the trihamiltonian recurrence (1), the Poisson tensor S is in a sense redundant: all the vector fields X_k can be obtained using just the two tensors P and Q. The paper [6] proposes a different kind of trihamiltonian recurrence: the Hamiltonians are not organized in a chain (labelled by one index only) but in a two-dimensional scheme (labelled by two indices) and satisfy the relations

$$P\mathrm{d}h_{k,j} = Q\mathrm{d}h_{k+1,j} = S\mathrm{d}h_{k,j+1}. \tag{2}$$

Clearly the two-dimensional scheme (2) can be thought of as a set of one-dimensional chains associated to the bihamiltonian pair P and Q, linked together by the third Poisson structure S. In this framework the third structure is no longer redundant, but allows one to construct new Hamiltonians starting from a given one. Moreover it is possible to collect Hamiltonians belonging to different chains in a unique recursion scheme. Some general properties of this kind of systems were investigated and a class of examples was constructed in [6,2], but until now very few systems were shown to be trihamiltonian in this sense.

The aim of this article is to construct a new Poisson structure on a subspace of the Kupershmidt algebra. This new Poisson tensor, together with the already known ones P and Q, leads to the construction of a trihamiltonian recurrence in the sense of (2), for an extension of the periodic Toda lattice. The presented construction extends to the n-particles case some partial results, obtained in [1,2] for the 3-particles case.

2. Background results

2.1. Poisson tensors and Hamiltonian vector fields

Liouville's definition of complete integrability concerns Hamiltonian systems of mechanical origin, whereby the phase space is a cotangent bundle endowed with the canonical Poisson bracket. However, it has been thoroughly clarified several decades ago that relevant families of integrable Hamiltonian systems are naturally defined in more general phase spaces, namely differentiable manifolds endowed with a (possibly degenerate) Poisson bracket, not derived from a natural symplectic structure but rather from other geometric or algebraic structures. A typical example is provided by the dual spaces of Lie algebras. We recall that a *Poisson tensor* (or equivalently a *Poisson structure*) on a manifold M is a bivector P (i.e. a skew-symmetric, contravariant, rank two tensor), such that the Schouten bracket $[P, P]$ vanishes. The latter condition ensures that the bracket defined by

$$\{f, g\} = P(\mathrm{d}f, \mathrm{d}g),$$

for an arbitrary pair of differentiable functions (f, g) on M fulfils the Jacobi identity and is therefore a Poisson bracket. A Poisson tensor P associates to any function h a corresponding *Hamiltonian vector field* X_h given by the expression

$$X_h = P\mathrm{d}h$$

but, in contrast to the case of the canonical Poisson brackets of cotangent bundles, the vector field X_h can be identically vanishing even if h is non-constant, because P can be degenerate. A function h such that $\mathrm{d}h \neq 0$ and $\mathrm{d}h \in \operatorname{Ker} P$ is called a *Casimir function* for the Poisson structure P and is in involution with any other function on M.

2.2. Bihamiltonian structures

As shown by Magri, there is a systematic connection between the complete integrability of a Hamiltonian vector field and the existence of a second, independent Poisson structure, that is preserved by the same vector field. Two different Poisson tensors P and Q are said to be *compatible* (in Magri's sense) if the linear combination (*pencil*) $Q - \lambda P$ is a Poisson tensor for any λ. A particular case of Poisson pencil is obtained if the Lie derivative of P along some vector field X is also a Poisson tensor: the tensor $Q = \mathcal{L}_X P$ is easily proved to be compatible with P. Two compatible Poisson tensors define a *bihamiltonian structure*, and a vector field is called *bihamiltonian* if it is Hamiltonian with respect to both Poisson tensors:

$$X_h = P\mathrm{d}h = Q\mathrm{d}\tilde{h}.$$

A *Lenard chain* is a set of bihamiltonian vector fields $X_k = P\mathrm{d}h_k$ such that the corresponding Hamiltonians satisfy the recursion relations

$$P\mathrm{d}h_k = Q\mathrm{d}h_{k+1}. \tag{3}$$

These recursion relations (3) can be represented by the diagram

$$\begin{array}{ccccccccc} & & X_1 & & \cdots & & X_k & & \\ & \overset{P}{\nearrow} & & \overset{Q}{\nwarrow}\ \overset{P}{\nearrow} & & \overset{Q}{\nwarrow}\ \overset{P}{\nearrow} & & \overset{Q}{\nwarrow} & \\ \mathrm{d}h_1 & & & \mathrm{d}h_2 & & \mathrm{d}h_k & & & \mathrm{d}h_{k+1} \end{array} \tag{4}$$

In the following we shall omit the indication of the differential acting on the Hamiltonian functions, and simply draw $h \overset{P}{\to} X_h$ to mean $P : \mathrm{d}h \mapsto X_h$. The recursion relations (3) imply that all the vector fields belonging to a Lenard chain mutually commute and hence their Hamiltonians are in involution with respect to both Poisson structures [16]. Lenard recursion is thus the most effective way to produce integrable systems on a bihamiltonian manifold, and it turns out that both finite-dimensional and infinite-dimensional classical examples of integrable systems (including the case of solitonic equations) can be obtained, together with their symmetries and first integrals, as Lenard chains for suitable bihamiltonian structures.

Whenever one of the Poisson tensors is degenerate constructing a Lenard chain by direct iteration becomes harder, but in that case one can look for a *Casimir function of the Poisson pencil*, i.e. a function f_λ, non-constant on M and depending (as a formal power series) on the parameter λ, such that $(Q - \lambda P)\mathrm{d}f_\lambda \equiv 0$. It is straightforward to check that all the coefficients of the power expansion of f_λ with respect to λ satisfy the recursion relations (3) and thus are in involution. If a Poisson pencil admits two different Casimir functions, $f_\lambda = \sum f_k \lambda^k$ and $g_\lambda = \sum g_j \lambda^j$, then the functions f_k are not necessarily in involution with the functions g_j. In this case, a sufficient condition for involution between the functions f_k and g_j (with respect to both Poisson structures) is the presence in the power expansions of f_λ or g_λ of a Casimir function for one of the Poisson structures.

2.3. Trihamiltonian structures

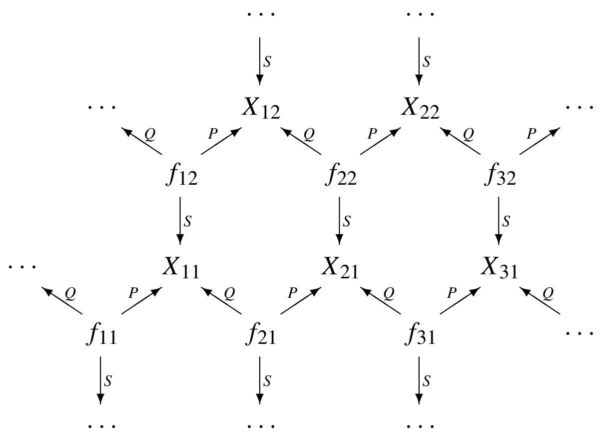
Although a pair of compatible Poisson structures provides a complete framework for the construction (or the investigation) of integrable systems, it has been recently suggested [6] that the existence of a third Poisson structure may be related to further properties, in turn connected to the *algebraic integrability*. A *trihamiltonian structure* on a manifold is given by three mutually compatible Poisson structures P, Q and S. A *common Casimir function* for a trihamiltonian structure is a function $f_{\lambda\mu}$ (dependent on two parameters) such that

$$\begin{cases} (Q - \lambda P)\mathrm{d}f_{\lambda\mu} = 0 \\ (S - \mu P)\mathrm{d}f_{\lambda\mu} = 0. \end{cases} \tag{5}$$

As a matter of fact, a generic trihamiltonian structure may not admit at all such a function (see [6] for a simple example). Although a precise classification of trihamiltonian structures admitting common Casimir functions is an open problem at present, the main point of introducing a third Poisson structure is exactly the possible existence of nontrivial (and complete, in a suitable sense) solutions to (5). If the function $f_{\lambda\mu}$ exists, the coefficients of its power expansion $f_{\lambda\mu} = \sum f_{kj}\lambda^k\mu^j$ obey the recursion relation

$$X_{kj} = P\mathrm{d}f_{kj} = Q\mathrm{d}f_{k+1,j} = S\mathrm{d}f_{k,j+1}, \tag{6}$$

which can be represented by the following recursion diagram:



Similarly to the bihamiltonian case, the recursion relations (6) imply that the coefficients f_{kj} are mutually in involution with respect to each of the three Poisson structures [6]. The vector fields $X_{kj} = P\mathrm{d}f_{kj} = Q\mathrm{d}f_{k+1,j} = S\mathrm{d}f_{k,j+1}$ are trihamiltonian vector fields.

It is worth observing that the power expansion of a common Casimir function does not contain negative powers (i.e. it is a formal Taylor series in both λ and μ) only if Q and S are degenerate Poisson tensors: for every power of λ the least order coefficient in μ must be a Casimir function for S, and the least order coefficient in λ must be a Casimir function for Q for every power of μ.

Moreover the apparently different role played by P in Eq. (5) depends only on the choice of the parameters λ and μ. In fact, setting $\lambda = \nu_P/\nu_Q$ and $\mu = \nu_P/\nu_S$ in (5), it is clear that every choice of two of the following three equations

$$\begin{cases} (\nu_Q Q - \nu_P P)\mathrm{d}f = 0 \\ (\nu_S S - \nu_P P)\mathrm{d}f = 0 \\ (\nu_Q Q - \nu_S S)\mathrm{d}f = 0 \end{cases}$$

implies the remaining one.

A remarkable feature of trihamiltonian systems is that, if a suitable set of vector fields Z_α is found, the common Casimir function allows one to calculate a set of separation coordinates for all its coefficients. The technical conditions under which this result holds are stated in the following proposition (see [6] for a proof):

Proposition 1. *Let P, Q and S be three mutually compatible Poisson structures of constant corank m on a manifold of dimension $2n + m$, let $f_{\lambda\mu}$ be a polynomial common Casimir function of the two pencils $Q - \lambda P$ and $S - \mu P$ containing m Casimir functions c^α of P and let h^α, and k^α be the coefficients of $f_{\lambda\mu}$ such that $P\mathrm{d}h^\alpha = Q\mathrm{d}c^\alpha$ and $P\mathrm{d}k^\alpha = S\mathrm{d}c^\alpha$ respectively. Given m independent vector fields Z_α let the following conditions hold:*

- $Z_\alpha(c^\beta) = \delta_\alpha^\beta$ *and* $[Z_\alpha, Z_\beta] = 0$;
- *if* $Q_d = Q + L_{h^\alpha Z_\alpha}P$ *and* $S_d = S + L_{k^\alpha Z_\alpha}P$ *then* $L_{Z_\alpha}P = L_{Z_\alpha}Q_d = L_{Z_\alpha}S_d = 0$;
- $Z_\alpha(Z_\beta(f_{\lambda\mu})) = 0$;
- *the polynomials* $\sigma_\alpha = Z_\alpha(f_{\lambda\mu})$ *have* $2n$ *functionally independent common roots* $\{\lambda_k, \mu_k\}$;

• *the equality*

$$\{\sigma_\alpha, \sigma_\beta\}_P\big|_{\lambda_k,\mu_k} = \left(\frac{\partial\sigma_\alpha}{\partial\lambda}\frac{\partial\sigma_\beta}{\partial\mu} - \frac{\partial\sigma_\beta}{\partial\lambda}\frac{\partial\sigma_\alpha}{\partial\mu}\right)\Big|_{\lambda_k,\mu_k}$$

holds and $\forall k\ \exists$ a pair (α, β) *for which both sides do not vanish identically.*

Then, on any symplectic leaf of P, the $2n$ functions $\{\lambda_k, \mu_k\}$ are canonical coordinates and there exists m (not necessarily distinct) polynomials $p_k(\lambda, \mu)$, with constant coefficients, such that

$$f_{\lambda_k\mu_k} = p_k(\lambda_k, \mu_k).$$

Hence the m functions $W_k = f_{\lambda\mu} - p_k(\lambda, \mu)$ establish the separation in Sklyanin sense of all the coefficients of $f_{\lambda\mu}$ in the coordinates $\{\lambda_k, \mu_k\}$.

2.4. Hamiltonian systems on Lie algebras

A particularly important example of a Poisson manifold is given by any Lie algebra $\mathfrak{g}$ equipped with a scalar product $(\cdot,\cdot)$. In this case, it is well known that given the functions f, g on $\mathfrak{g}$ and their gradient $\nabla f, \nabla g$ with respect to $(\cdot,\cdot)$, the bracket

$$\{f, g\}(L) = (L, [\nabla f, \nabla g])$$

is a Poisson bracket called the *Lie–Poisson bracket*. An analogous Poisson bracket is defined on the dual $\mathfrak{g}^*$ of the Lie algebra. In this paper, the scalar product on $\mathfrak{g}$ is supposed to be *invariant*, i.e. for every $A, B, C \in \mathfrak{g}$ to satisfy

$$(A, [B, C]) = (B, [C, A]).$$

A r-matrix [26,24] on a Lie algebra is a linear map $R : \mathfrak{g} \to \mathfrak{g}$ satisfying the Yang–Baxter equation

$$[R(A), R(B)] - R([R(A), B] + [A, R(B)]) = -[A, B]. \tag{7}$$

If the Lie bracket $[\cdot,\cdot]$ is the commutator obtained from an underlying associative algebra structure, and if the r-matrix satisfies a suitable condition, a bihamiltonian structure can be defined. Indeed, if $\mathfrak{g}$ is an associative algebra, the following two natural operations are defined:

$$[A, B] = AB - BA; \tag{8}$$

$$A \bullet B = \frac{AB + BA}{2}. \tag{9}$$

It is worth observing that the product $\bullet$ is commutative, but in general not associative, and satisfies the relation

$$[A, B \bullet C] = [A, B] \bullet C + B \bullet [A, C]. \tag{10}$$

This richer structure allows, as proved in [23], to state a sufficient condition for the existence of a bihamiltonian structure on $\mathfrak{g}$:

Proposition 2. *Given a linear map $R : \mathfrak{g} \to \mathfrak{g}$ with its transpose R^{t} defined through $(A, R(B)) = (R^{\mathrm{t}}(A), B)$, if both R and $R_a = (R - R^{\mathrm{t}})/2$ satisfy the Yang–Baxter equation (7) then the two brackets*

$$\{f, g\}_{\mathrm{Sem}}(L) = (L, [R(\nabla f), \nabla g] + [\nabla f, R(\nabla g)]) \tag{11}$$

$$\{f, g\}_{\mathrm{Skl}}(L) = (L, [R(L \bullet \nabla f), \nabla g] + [\nabla f, R(L \bullet \nabla g)]) \tag{12}$$

are two compatible Poisson brackets on $\mathfrak{g}$.

The linear Poisson structure (11) and the quadratic one (12) will be called respectively the *Semënov–Tian–Shansky bracket* and *Sklyanin bracket*. An important particular case of r-matrix (the so-called *split case* [24]) is obtained if the Lie algebra $\mathfrak{g}$ is the direct sum of two Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$. Given an invariant scalar product, the orthogonal decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ with $\mathfrak{g}^\pm = (\mathfrak{g}_\pm)^\perp$ can be constructed. The map

$$R = \Pi_+ - \Pi_-$$

is thus a r-matrix and, if it satisfies the condition of Proposition 2, the Hamiltonian vector fields generated by the two structures (11) and (12) can be set in both the two forms

$$X_f^{\text{Sem}}(L) = [L, (\nabla f)_\pm] - [L, \nabla f]^\pm \tag{13}$$

$$X_f^{\text{Skl}}(L) = [L, (L \bullet \nabla f)_\pm] - L \bullet [L, \nabla f]^\pm. \tag{14}$$

A remark on the Lie derivative of a Lie–Poisson bracket ends this introductory section:

Lemma 3. *Let X be a vector field on a Lie algebra $\mathfrak{g}$ and P be the Poisson tensor associated to the Lie–Poisson bracket. Then the expression for the bracket (not necessarily a Poisson one) associated to $\mathcal{L}_X P$ is (omitting the evaluation point L):*

$$\{f, g\}_{\mathcal{L}_X P} = (X, [F, G]) - (L, [X'^{\text{t}} \cdot F, G] + [F, X'^{\text{t}} \cdot G]),$$

where $X' \cdot M$ is the derivative of X (thought of as a function from the vector space $\mathfrak{g}$ to itself) in the direction of the vector M, $F = \nabla f$ and $G = \nabla g$.

Proof. The expression of the Poisson bracket associated to the Lie derivative of a Poisson tensor is:

$$\{f, g\}_{\mathcal{L}_X P} = X(\{f, g\}_P) - \{X(f), g\}_P - \{f, X(g)\}_P.$$

Considering the Lie–Poisson structure one has

$$\begin{aligned} X(\{f, g\}_P)(L) &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} (L + \tau X, [F(L + \tau X), G(L + \tau X)]) \\ &= (X, [F, G]) + (L, [F' \cdot X, G]) + (L, [F, G' \cdot X]) \end{aligned}$$

and, because $X(f) = (X, F)$,

$$\begin{aligned} (\nabla X(f), M) &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} (X(L + \tau M), F(L + \tau M)) \\ &= (X, F' \cdot M) + (X' \cdot M, F) = (M, F'^{\text{t}} \cdot X + X'^{\text{t}} \cdot F) \end{aligned}$$

thus $\nabla X(f) = F'^{\text{t}} \cdot X + X'^{\text{t}} \cdot F$. Because $F = \nabla f$, one obtains

$$\begin{aligned} (F'^{\text{t}} \cdot M, N) = (M, F' \cdot N) &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} (M, F(L + \tau N)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} f(L + \tau N + \sigma M) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} (N, F(L + \sigma M)) = (N, F' \cdot M) \end{aligned}$$

hence the relation $F'^{\text{t}} \cdot M = F' \cdot M$ holds. On the other hand, note that $X'^{\text{t}} \cdot M$ is not generally equal to $X' \cdot M$. A simple substitution ends the proof. □

3. A new trihamiltonian structure

3.1. The Kupershmidt algebra

The Kupershmidt algebra $\mathcal{K}_n$ is an associative algebra with unity introduced in [14] in order to establish an algebraic framework for the periodic Toda lattice. The elements of $\mathcal{K}_n$ are polynomials in Δ and Δ^{-1} with coefficients in the ring (with componentwise operations) of n-periodic sequences $\sigma = (\sigma^{(1)}, \ldots, \sigma^{(n)})$: an element of $\mathcal{K}_n$ has the form $\sum \sigma_k \Delta^k$. The multiplication between powers of Δ is commutative and given by $\Delta^l \Delta^k = \Delta^{l+k}$, the identity is Δ^0 and the multiplication between powers of Δ and sequences is defined by the formula

$$\Delta^k \sigma = \tau \Delta^k,$$

$$\sigma \longrightarrow \begin{pmatrix} \sigma^{(1)} & 0 & \cdots & 0 \\ 0 & \sigma^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^{(n)} \end{pmatrix}$$

$$\sigma\Delta \longrightarrow \begin{pmatrix} 0 & \sigma^{(1)} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma^{(n-1)} \\ \lambda\sigma^{(n)} & 0 & \cdots & 0 \end{pmatrix} \qquad \Delta^{-1}\sigma \longrightarrow \begin{pmatrix} 0 & \cdots & 0 & \lambda^{-1}\sigma^{(n)} \\ \sigma^{(1)} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \sigma^{(n-1)} & 0 \end{pmatrix}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\sigma\Delta^{n-1} \longrightarrow \begin{pmatrix} 0 & \cdots & 0 & \sigma^{(1)} \\ \lambda\sigma^{(2)} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda\sigma^{(n)} & 0 \end{pmatrix} \qquad \Delta^{-n+1}\sigma \longrightarrow \begin{pmatrix} 0 & \lambda^{-1}\sigma^{(2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \lambda^{-1}\sigma^{(n)} \\ \sigma^{(1)} & 0 & \cdots & 0 \end{pmatrix}$$

$$\sigma\Delta^{n} \longrightarrow \begin{pmatrix} \lambda\sigma^{(1)} & 0 & \cdots & 0 \\ 0 & \lambda\sigma^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda\sigma^{(n)} \end{pmatrix} \qquad \Delta^{-n}\sigma \longrightarrow \begin{pmatrix} \lambda^{-1}\sigma^{(1)} & 0 & \cdots & 0 \\ 0 & \lambda^{-1}\sigma^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda^{-1}\sigma^{(n)} \end{pmatrix}$$

Fig. 1. Isomorphism between the algebras $\mathcal{K}(n)$ and $\mathfrak{gl}(n)[\lambda, \lambda^{-1}]$.

where the sequence τ is given by

$$\tau = (\sigma^{(k+1)}, \ldots, \sigma^{(n)}, \sigma^{(1)}, \ldots, \sigma^{(k)}).$$

Hence, Δ can be interpreted as a shift operator on n-periodic sequences.

In [21] the algebra $\mathcal{K}_n$ was shown to be isomorphic, through the map shown in Fig. 1, to the algebra $\mathfrak{gl}(n)[\lambda, \lambda^{-1}]$ of polynomials in λ and λ^{-1} with matrix coefficients: an element of $\mathfrak{gl}(n)[\lambda, \lambda^{-1}]$ has the form $L_\lambda = \sum L_k \lambda^k$. On the algebra $\mathcal{K}_n$ one can define the linear involution $*$ by $*(\sigma\Delta^k) = \Delta^{-k}\sigma$ and the trace

$$\mathfrak{Tr}\left(\sum \sigma_k \Delta^k\right) = \sum_{j=1}^{n} \sigma_0^{(j)};$$

under the isomorphism between $\mathcal{K}_n$ and $\mathfrak{gl}(n)[\lambda, \lambda^{-1}]$ they are mapped in the involution $*(L_\lambda) = (L_{\lambda^{-1}})^t$ and in the trace

$$\mathfrak{Tr}\left(\sum L_k \lambda^k\right) = \mathrm{tr}(L_0).$$

From now on the two algebras $\mathcal{K}_n$ and $\mathfrak{gl}(n)[\lambda, \lambda^{-1}]$ will be identified and indicated with $\mathfrak{G}$. The representation used will be clear from the context.

On the associative algebra $\mathfrak{G}$ one can define the commutator (8), the commutative product (9) and the invariant scalar product

$$(L, M) = \mathfrak{Tr}(LM). \tag{15}$$

Moreover, it is possible to find a decomposition of $\mathfrak{G}$ into two Lie subalgebras and thus a r-matrix: following [21] one introduces the subspaces

- $\mathfrak{G}_+$: skewsymmetric elements, $*L = -L$;
- $\mathfrak{G}^+$: symmetric elements, $*L = L$;
- $\mathfrak{G}_k$: elements of degree k, with powers of Δ not greater than k;
- $\mathfrak{G}_- = \mathfrak{G}_0$;
- $\mathfrak{G}^- = \mathfrak{G}_{-1}$.

As is easily seen, $\mathfrak{G}_\pm$ are two subalgebras of $\mathfrak{G}$, $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_- = \mathfrak{G}^+ \oplus \mathfrak{G}^-$ and $\mathfrak{G}^\pm = (\mathfrak{G}_\pm)^\perp$. The linear map $R = \Pi_+ - \Pi_-$ is thus a r-matrix and its skewsymmetric part $(R - R^{\mathrm{t}})/2$ also satisfies the Yang–Baxter equation [21]. Proposition 2 implies that the two brackets (11) and (12) endow $\mathfrak{G}$ with a bihamiltonian structure and the corresponding Hamiltonian vector fields can be set respectively in the form (13) and (14).

A Poisson structure is said to be *reducible by restriction* on a given submanifold if any Hamiltonian vector field is tangent to the submanifold, when evaluated on it. In other words, if the submanifold is invariant with respect to any Hamiltonian vector field. On the algebra $\mathfrak{G}$ a reduction procedure for the Poisson structures (11) and (12) can be performed: the bihamiltonian structure of $\mathfrak{G}$ is reducible on $\mathfrak{G}_k^+ = \mathfrak{G}^+ \cap \mathfrak{G}_k$ for any k. This is the content of the following theorem, explicitly proved in [21] for the case $\mathfrak{G}_1^+$ and somehow already present in [23].

Theorem 4. *Given any function $f : \mathfrak{G} \to \mathbb{R}$, both the subspaces $\mathfrak{G}^+$ and $\mathfrak{G}_k$ are invariant with respect to both the corresponding Hamiltonian vector fields (13) and (14). As a consequence, the two Poisson structures (11) and (12) are reducible by restriction on $\mathfrak{G}^+$, $\mathfrak{G}_k$ and on their intersection $\mathfrak{G}_k^+$.*

Proof. Firstly, the reducibility of the Semënov–Tian–Shansky structure is proved. Set $F = \nabla f$, the invariance of $(\cdot,\cdot)$ implies

$$(M, [L, F_+]) = (L, [F_+, M])$$

and being $\mathfrak{G}_+$ a subalgebra orthogonal to $\mathfrak{G}^+$, if $M \in \mathfrak{G}_+$ and $L \in \mathfrak{G}^+$ then

$$(M, [L, F_+]) = 0 \quad \forall M \in \mathfrak{G}_+ \;\Rightarrow\; [L, F_+] \in \mathfrak{G}^+.$$

From the expression (13) one gets that $X_f^{\mathrm{Sem}} \in \mathfrak{G}^+$. On the other hand, if $L \in \mathfrak{G}_k$ then $[L, F_-]$ has maximum degree k, because $F_- \in \mathfrak{G}_0$, and $[L, F]^- \in \mathfrak{G}_{-1}$. From (13) one gets that $X_f^{\mathrm{Sem}} \in \mathfrak{G}_k$.

Analogously, from (14) one can prove the reducibility of the Sklyanin Poisson structure. Observing that, if $L \in \mathfrak{G}^+$, then $[L, (L \bullet \nabla f)_+] \in \mathfrak{G}^+$, and that $\mathfrak{G}^+ \bullet \mathfrak{G}^+ \subset \mathfrak{G}^+$ one obtains that $X_f^{\mathrm{Skl}} \in \mathfrak{G}^+$. Moreover, because $L \in \mathfrak{G}_k$ either $[L, (L \bullet \nabla f)_-]$ or $L \bullet [L, \nabla f]^-$ has maximum degree k.

The reducibility of the two Poisson structures on the intersection $\mathfrak{G}_k^+ = \mathfrak{G}^+ \cap \mathfrak{G}_k$ follows straightforwardly. □

3.2. The third Hamiltonian structure

The trihamiltonian structure proposed in this paper is defined on the subspace $\mathfrak{G}_{n-1}^+$ of the algebra $\mathfrak{G}$. An element of $\mathfrak{G}_{n-1}^+$ can be expressed in one of the two forms:

$$\Delta^{n-1}\sigma_{n-1} + \cdots + \Delta\sigma_1 + \sigma_0 + \sigma_1\Delta + \cdots + \sigma_{n-1}\Delta^{n-1}$$
$$L_0 + L_1\lambda + L_1^t\lambda^{-1}$$

where L_0 is a symmetric matrix and L_1 is a strictly lower triangular matrix (both L_0 and L_1 do not depend on λ).

Let us consider the homomorphism Φ, between the vector space $\mathfrak{G}$ and the vector space $\mathfrak{gl}(n, \mathbb{C})$, defined as

$$\begin{aligned} &\Phi : \mathfrak{G} \to \mathfrak{gl}(n, \mathbb{C}) \\ &L_\lambda \mapsto L = L_{\lambda=i}. \end{aligned} \tag{16}$$

The restriction of Φ to the subspace $\mathfrak{G}_{n-1}^+$ is invertible, its image is the set $\mathfrak{H}$ of Hermitian matrices. Given $L_\lambda = L_0 + L_1\lambda + L_1^t\lambda^{-1} \in \mathfrak{G}_{n-1}^+$ one has

$$L = \Phi(L_\lambda) = L_0 + i(L_1 - L_1^t) = L_S + iL_A,$$

where L_S is a real symmetric matrix while L_A is a real skewsymmetric one.

The set of Hermitian matrices $\mathfrak{H}$ can be equipped with a Lie bracket $[\![\cdot,\cdot]\!]$, a commutative product $\odot$ and a scalar product $(\!(\cdot,\cdot)\!)$ in the following way:

$$[\![L, M]\!] = i(LM - ML) = i([L_S, M_S] - [L_A, M_A]) - [L_S, M_A] - [L_A, M_S] \tag{17}$$

$$L \odot M = \frac{1}{2}(LM + ML) = L_S \bullet M_S - L_A \bullet M_A + i(L_A \bullet M_S + L_S \bullet M_A) \tag{18}$$

$$(\!(L, M)\!) = \mathrm{tr}(LM) = \mathrm{tr}(L \odot M) = \mathrm{tr}(L_S M_S - L_A M_A) \tag{19}$$

where the standard Lie bracket and commutative product (8) and (9) on $\mathfrak{gl}(n)$ are used. From these definitions, because the trace of the product of two Hermitian matrices is always real and the diagonal entries of the product of two elements of $\mathfrak{G}^+_{n-1}$ does not depend on λ, the following two important lemmas can be proved by straightforward computations:

Lemma 5. *If $L_\lambda, M_\lambda \in \mathfrak{G}^+_{n-1}$ then the map Φ satisfies the following relations:*

$$\mathfrak{Tr}(L_\lambda) = \mathrm{tr}(L), \qquad (L_\lambda, M_\lambda) = (\!(L, M)\!),$$
$$\Phi([L_\lambda, M_\lambda]) = -i[\![L, M]\!], \qquad \Phi(L_\lambda \bullet M_\lambda) = L \odot M.$$

Lemma 6. *The Lie bracket (17), the commutative product (18) and the scalar product (19) on $\mathfrak{H}$ satisfies the following properties:*

$$(\!(L \odot M, N)\!) = (\!(L, M \odot N)\!), \qquad (\!(L, [\![M, N]\!])\!) = (\!(M, [\![N, L]\!])\!),$$
$$[\![L, M \odot N]\!] = [\![L, M]\!] \odot N + M \odot [\![L, N]\!].$$

The previous construction needs some remarks. First, the set of Hermitian matrices with the Lie bracket (17) is isomorphic (through the map $\Psi : L \mapsto -iL$) to the Lie algebra of skew-Hermitian matrices with the ordinary commutator, the choice of Hermitian matrices allows a simpler form for the map Φ. Second, $\Psi \circ \Phi$ is a homomorphism between the associative algebras $\mathfrak{G}$ and $\mathfrak{gl}(n, \mathbb{C})$ but, even when restricted to $\mathfrak{G}^+_{n-1}$, it is not an isomorphism being $\mathfrak{G}^+_{n-1}$ only a subspace and not an associative subalgebra of $\mathfrak{G}$.

In order to rewrite on $\mathfrak{H}$ the restrictions on $\mathfrak{G}^+_{n-1}$ of the Sklyanin and Semënov–Tian–Shansky brackets it is necessary to introduce the map

$$\rho : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$$
$$L \mapsto L^\uparrow - L^\downarrow$$

(where $L^\uparrow$, $L^\downarrow$ are respectively the strictly upper and the lower triangular part of the matrix $L = L^d + L^\uparrow + L^\downarrow$) and the two maps

$$\mathtt{a} : \mathfrak{H} \to \mathfrak{H} \qquad \mathtt{s} : \mathfrak{H} \to \mathfrak{H}$$
$$L \mapsto i\rho(L_S) \qquad L \mapsto \rho(L_A).$$

It is well known that the map ρ is a r-matrix on $\mathfrak{gl}(n, \mathbb{C})$ with the standard commutator; the maps $\mathtt{a}$ and $\mathtt{s}$ are each the transpose of the other with respect to $(\!(\cdot,\cdot)\!)$:

Lemma 7. *The maps* $\mathtt{a}$ *and* $\mathtt{s}$ *satisfy the relation*

$$(\!(\mathtt{s}(L), M)\!) = (\!(L, \mathtt{a}(M))\!).$$

Proof.

$$\begin{aligned}
(\!(\mathtt{s}(L), M)\!) = \mathrm{tr}[\rho(L_A) M_S] &= \mathrm{tr}\left[(L_A^\uparrow - L_A^\downarrow)(M_S^d + M_S^\uparrow + M_S^\downarrow)\right] \\
&= \mathrm{tr}(L_A^\uparrow M_S^\downarrow - L_A^\downarrow M_S^\uparrow), \\
(\!(L, \mathtt{a}(M))\!) = -\mathrm{tr}[L_A \rho(M_S)] &= \mathrm{tr}\left[(L_A^\uparrow + L_A^\downarrow)(M_S^\downarrow - M_S^\uparrow)\right] \\
&= \mathrm{tr}(L_A^\uparrow M_S^\downarrow - L_A^\downarrow M_S^\uparrow).
\end{aligned}$$

The vector space isomorphism Φ allows one to write on $\mathfrak{H}$ the (restrictions of) Sklyanin and Semënov–Tian–Shansky brackets defined on $\mathfrak{G}^+_{n-1}$. □

Proposition 8. *The Sklyanin Poisson bracket, rewritten on $\mathfrak{H}$ through the isomorphism (16), has the expression:*

$$\{f \circ \Phi, g \circ \Phi\}_{\text{Skl}}(L_\lambda) = ((L \odot \mathtt{s}(L), [\![F, G]\!])) - ((L, [\![F \odot \mathtt{s}(L) + \mathtt{a}(L \odot F), G]\!] + [\![F, G \odot \mathtt{s}(L) + \mathtt{a}(L \odot G)]\!])) \tag{20}$$

where f, g are two functions $\mathfrak{H} \to \mathbb{R}$ and F, G their gradients with respect to $((\cdot,\cdot))$.

The proof of this statement can be found in the Appendix; the corresponding expression for the Semënov–Tian–Shansky bracket on $\mathfrak{H}$ immediately follows from the fact that

$$\{\tilde{f}, \tilde{g}\}_{\text{Skl}}(L_\lambda - \mu\mathbb{1}) = \{\tilde{f}, \tilde{g}\}_{\text{Skl}}(L_\lambda) - \mu\{\tilde{f}, \tilde{g}\}_{\text{Sem}}(L_\lambda).$$

Proposition 9. *The Semënov–Tian–Shansky Poisson bracket, rewritten on $\mathfrak{H}$ through the isomorphism (16), has the expression:*

$$\{f \circ \Phi, g \circ \Phi\}_{\text{Sem}}(L_\lambda) = ((L, \mathtt{a}([\![F, G]\!]) - [\![\mathtt{a}(F), G]\!] - [\![F, \mathtt{a}(G)]\!])). \tag{21}$$

Proof. Substituting $L_\lambda - \mu\mathbb{1}$ in (20) and observing that $\mathtt{s}(\mathbb{1}) = 0$ one obtains

$$\begin{aligned} &(((L - \mu\mathbb{1}) \odot \mathtt{s}(L), [\![F, G]\!])) - ((L - \mu\mathbb{1}, [\![F \odot \mathtt{s}(L) + \mathtt{a}[(L - \mu\mathbb{1}) \odot F], G]\!] \\ &\quad + [\![F, G \odot \mathtt{s}(L) + \mathtt{a}[(L - \mu\mathbb{1}) \odot G]]\!])) \\ &\ = \{f \circ \Phi, g \circ \Phi\}_{\text{Skl}}(L_\lambda) - \mu((\mathtt{s}(L), [\![F, G]\!])) + \mu((L, [\![\mathtt{a}(F), G]\!] + [\![F, \mathtt{a}(G)]\!])). \end{aligned}$$

Thus, from

$$((\mathtt{s}(L), [\![F, G]\!])) = ((L, \mathtt{a}([\![F, G]\!])))$$

the thesis follows. □

The third bracket of the trihamiltonian structure on $\mathfrak{G}^+_{n-1} \simeq \mathfrak{H}$ is simply the Lie–Poisson bracket associated to the Lie bracket (17):

$$\{f, g\}_{\text{Lin}}(L) = ((L, [\![F, G]\!])). \tag{22}$$

Theorem 10. *The three brackets $\{\cdot,\cdot\}_{\text{Sem}}$, $\{\cdot,\cdot\}_{\text{Lin}}$ and $\{\cdot,\cdot\}_{\text{Skl}}$ are mutually compatible Poisson structures, hence they endow $\mathfrak{H}$ (and consequently $\mathfrak{G}^+_{n-1}$) with a trihamiltonian structure.*

Proof. The two brackets $\{\cdot,\cdot\}_{\text{Sem}}$ and $\{\cdot,\cdot\}_{\text{Skl}}$ are compatible Poisson brackets on $\mathfrak{H}$ because they are induced by compatible Poisson brackets on $\mathfrak{G}^+_{n-1} \subset \mathfrak{G}$, and the bracket $\{\cdot,\cdot\}_{\text{Lin}}$ is a Poisson bracket because it is of Lie–Poisson type. It remains to prove that $\{\cdot,\cdot\}_{\text{Lin}}$ is compatible with both the others.

Considering the two vector fields

$$X(L) = \mathtt{s}(L), \qquad Y(L) = L \odot \mathtt{s}(L)$$

one observes that

$$\begin{aligned} X' \cdot M &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \mathtt{s}(L + \tau M) = \mathtt{s}(M) \\ Y' \cdot M &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} (L + \tau M) \odot \mathtt{s}(L + \tau M) = L \odot \mathtt{s}(M) + M \odot \mathtt{s}(L) \end{aligned}$$

and thus (using Lemma 7)

$$\begin{aligned}((X'\cdot M,N)) &= ((\mathtt{s}(M),N)) = ((M,\mathtt{a}(N)))\\ ((Y'\cdot M,N)) &= ((L\odot \mathtt{s}(M)+M\odot \mathtt{s}(L),N)) = ((M,N\odot \mathtt{s}(L)))+((\mathtt{s}(M),L\odot N))\\ &= ((M,N\odot \mathtt{s}(L)+\mathtt{a}(L\odot N))).\end{aligned}$$

As a consequence $X'^{\mathrm{t}}\cdot N=\mathtt{a}(N)$ and $Y'^{\mathrm{t}}\cdot N=N\odot \mathtt{s}(L)+\mathtt{a}(L\odot N)$. Lemma 3 therefore implies that both the Semënov–Tian–Shansky and the Sklyanin tensors are the Lie derivative of the Poisson tensor associated to $\{\cdot,\cdot\}_{\mathrm{Lin}}$, respectively through the vector fields X and Y. The two structures $\{\cdot,\cdot\}_{\mathrm{Sem}}$ and $\{\cdot,\cdot\}_{\mathrm{Skl}}$ are thus compatible with the new structure $\{\cdot,\cdot\}_{\mathrm{Lin}}$. □

4. Application to the periodic Toda lattice

The aim of this section is to apply the trihamiltonian structure given in the previous section to the construction of a trihamiltonian extension of the periodic Toda lattice. The extended system has a richer geometrical structure than the ordinary one: indeed, only two out of the three Poisson structures can be reduced on the subspace on which the ordinary periodic Toda lattice is defined, producing its well known bihamiltonian formulation. The new linear structure (22) is not reducible.

The periodic n-particle Toda lattice has been extensively investigated and a wide literature has been produced on this subject. Here just a few basic facts are recalled in order to fix the notation and to allow the application of the general trihamiltonian construction. The exposition is largely based on the paper [21] to which the reader is referred for details and proofs; see also [23,1].

4.1. Trihamiltonian extension

The first step in the study of the Toda lattice is usually the introduction of the so-called Flaschka coordinates

$$a_k=\frac{1}{2}\mathrm{e}^{\frac{1}{2}(x_k-x_{k+1})},\qquad b_k=-\frac{1}{2}p_k.$$

In these coordinates the evolution equations for the periodic Toda lattice become

$$\dot a_k=a_k(b_{k+1}-b_k),\qquad \dot b_k=2(a_k^2-a_{k-1}^2)\quad k=1\ldots n \tag{23}$$

where, as usual, the periodicity condition $a_{k+n}=a_k$, $b_{k+n}=b_k$ is assumed.

Choosing the Lax pair

$$L_\lambda=\begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_n/\lambda\\ a_1 & b_2 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & & \ddots & \ddots & a_{n-1}\\ a_n\lambda & 0 & \cdots & a_{n-1} & b_n\end{pmatrix}\qquad B_\lambda=\begin{pmatrix} 0 & a_1 & 0 & \cdots & -a_n/\lambda\\ -a_1 & 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & & \ddots & \ddots & a_{n-1}\\ a_n\lambda & 0 & \cdots & -a_{n-1} & 0\end{pmatrix} \tag{24}$$

the Eq. (23) can be put into Lax form (with spectral parameter):

$$\frac{\mathrm{d}}{\mathrm{d}t}L_\lambda=[L_\lambda,B_\lambda]. \tag{25}$$

The matrix L_λ in Eq. (25) is easily recognized as an element of the subspace $\mathfrak{G}_1^+$, with $\sigma_0=(b_1,\ldots,b_n)$ and $\sigma_1=(a_1,\ldots,a_n)$. The Poisson tensors $\widehat{P}_{\mathrm{Sem}}$ and $\widehat{P}_{\mathrm{Skl}}$, obtained from the reduction of the two structures (11) and (12) on $\mathfrak{G}_1^+$, allow one to write the vector field corresponding to the Lax equations (25) in a bihamiltonian way:

$$X_{\text{Toda}} = \widehat{P}_{\text{Sem}} \mathrm{d}\widehat{H}_{\text{Sem}} = \widehat{P}_{\text{Skl}} \mathrm{d}\widehat{H}_{\text{Skl}} \tag{26}$$

where the Hamiltonians are given by

$$\widehat{H}_{\text{Skl}} = \mathfrak{Tr}(L_\lambda), \qquad \widehat{H}_{\text{Sem}} = \frac{\mathfrak{Tr}(L_\lambda^2) - (\mathfrak{Tr}(L_\lambda))^2}{2}.$$

Taking two Hamiltonians H_{Sem} and H_{Skl}, on $\mathfrak{G}_k^+ \supset \mathfrak{G}_1^+$, such that

$$H_{Sem}|_{\mathfrak{G}_1} = \widehat{H}_{\text{Sem}}, \qquad H_{\text{Skl}}|_{\mathfrak{G}_1} = \widehat{H}_{\text{Skl}} \tag{27}$$

the bihamiltonian vector field associated to them by the (suitably reduced) Poisson structures (11) and (12) can be thought of as an extension of the periodic Toda lattice.

The following theorem states that there exists an extension X_{Ext} of the Toda vector field X_{Toda} that is trihamiltonian with respect to the three structures $\{\cdot,\cdot\}_{\text{Sem}}$, $\{\cdot,\cdot\}_{\text{Lin}}$ and $\{\cdot,\cdot\}_{\text{Skl}}$.

Theorem 11. *Indicated with P_{Sem}, P_{Lin} and P_{Skl} are the Poisson tensors corresponding respectively to the brackets $\{\cdot,\cdot\}_{\text{Sem}}$, $\{\cdot,\cdot\}_{\text{Lin}}$ and $\{\cdot,\cdot\}_{\text{Skl}}$, the recurrence relations*

$$P_{\text{Sem}} \mathrm{d}H_{\text{Sem}} = P_{\text{Lin}} \mathrm{d}H_{\text{Lin}} = P_{\text{Skl}} \mathrm{d}H_{\text{Skl}}$$

hold, where

$$H_{\text{Skl}} = \operatorname{tr}(L), \qquad H_{\text{Lin}} = -\operatorname{tr}(L \odot \mathtt{a}(L)), \qquad H_{\text{Sem}} = \frac{\operatorname{tr}(L^2) - (\operatorname{tr}(L))^2}{2}.$$

The vector field $X_{\text{Ext}} = P_{\text{Sem}} \mathrm{d}H_{\text{Sem}}$ is an extension of the Toda vector field (26) in the sense of Eq. (27).

Proof. Firstly the gradient of H_{Skl}, H_{Lin} and H_{Sem} with respect to the scalar product $(\!(\cdot,\cdot)\!)$ is needed. The simple calculations

$$(\!(\nabla \operatorname{tr}(L), M)\!) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \operatorname{tr}(L + \tau M) = (\!(\mathbb{1}, M)\!)$$

$$\begin{aligned} (\!(\nabla \operatorname{tr}(L \odot \mathtt{a}(L)), M)\!) &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \operatorname{tr}[(L+\tau M) \odot \mathtt{a}(L+\tau M)] \\ &= (\!(L, \mathtt{a}(M))\!) + (\!(M, \mathtt{a}(L))\!) = (\!(\mathtt{s}(L) + \mathtt{a}(L), M)\!) \end{aligned}$$

$$(\!(\nabla H_{\text{Sem}}, M)\!) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \frac{\operatorname{tr}\left[(L+\tau M)^2\right] - [\operatorname{tr}(L+\tau M)]^2}{2} = (\!(L - \operatorname{tr}(L)\mathbb{1}, M)\!)$$

give

$$\nabla H_{\text{Skl}} = \mathbb{1}, \qquad \nabla H_{\text{Lin}} = -\mathtt{s}(L) - \mathtt{a}(L), \qquad \nabla H_{\text{Sem}} = L - \operatorname{tr}(L)\mathbb{1}.$$

Then substituting the previous expressions for the gradient respectively in (20)–(22) one obtains, for any function g:

$$\begin{aligned} \{H_{\text{Lin}}, g\}_{\text{Lin}} &= -(\!(L, [\![\mathtt{s}(L) + \mathtt{a}(L), \nabla g]\!])\!), \\ \{H_{\text{Skl}}, g\}_{\text{Skl}} &= -(\!(L, [\![\mathtt{s}(L) + \mathtt{a}(L), \nabla g]\!])\!), \\ \{H_{\text{Sem}}, g\}_{\text{Sem}} &= (\!(L, \mathtt{a}([\![L, \nabla g]\!]) - [\![\mathtt{a}(L), \nabla g]\!] - [\![L, \mathtt{a}(\nabla g)]\!])\!) = -(\!(L, [\![\mathtt{s}(L) + \mathtt{a}(L), G]\!])\!). \end{aligned}$$

Thus the recursion relation holds. The identities (27) follow trivially from the definition and Lemma 5. □

A final observation on the choice of the Hamiltonians is necessary: it is also possible to obtain the vector fields X_{Ext} and X_{Toda} respectively with the two alternative Hamiltonians

$$H_{\mathrm{Sem}} = \frac{1}{2}\mathrm{tr}(L^2), \qquad \widehat{H}_{\mathrm{Sem}} = \frac{1}{2}\mathfrak{Tr}(L_\lambda^2)$$

but the choice of Theorem 11, as will be shown in the examples, allows one to find a recurrence scheme with a finite number of Hamiltonians.

4.2. Some examples

The ordinary periodic Toda lattice with 3 particles is set in the subspace $\mathfrak{G}_1^+$ of $\mathfrak{gl}(3)[\lambda, \lambda^{-1}]$. In Flaschka coordinates, (b_k, a_k) the formula for the Lax matrix (24), becomes

$$L = \Delta^{-1}a + b + a\Delta \equiv \begin{pmatrix} b_1 & a_1 & a_3/\lambda \\ a_1 & b_2 & a_2 \\ a_3\lambda & a_2 & b_3 \end{pmatrix}.$$

The extension is obtained considering the subspace $\mathfrak{G}_2^+$ with coordinates (b_k, a_k, c_k), corresponding to a Lax matrix

$$L = \Delta^{-2}c + \Delta^{-1}a + b + a\Delta + c\Delta^2 \equiv \begin{pmatrix} b_1 & a_1 + \dfrac{c_1}{\lambda} & c_3 + \dfrac{a_3}{\lambda} \\ a_1 + c_1\lambda & b_2 & a_2 + \dfrac{c_2}{\lambda} \\ c_3 + a_3\lambda & a_2 + c_2\lambda & b_3 \end{pmatrix}.$$

The three Poisson structures P_{Sem}, P_{Skl} and P_{Lin}, given respectively by (11), (12) and (22), are

$$P_{\mathrm{Sem}} = \begin{pmatrix} 0 & 0 & 0 & a_1 & 0 & -a_3 & c_1 & c_2 & 0 \\ & 0 & 0 & -a_1 & a_2 & 0 & 0 & c_2 & -c_3 \\ & & 0 & 0 & -a_2 & a_3 & -c_1 & 0 & c_3 \\ & & & 0 & c_1 & -c_3 & 0 & 0 & 0 \\ & & & & 0 & c_2 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & * & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix}$$

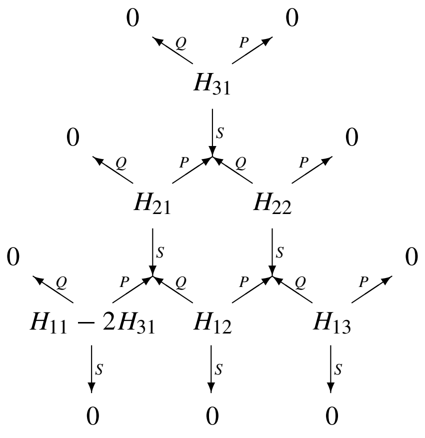
$$P_{\mathrm{Skl}} = \begin{pmatrix} 0 & 2(a_1^2 - c_2^2) & 2(c_1^2 - a_3^2) & b_1a_1 & 2(c_1a_1 - a_3c_2) & -b_1a_3 & b_1c_1 & -b_1c_2 & 0 \\ & 0 & 2(a_2^2 - c_3^2) & -b_2a_1 & b_2a_2 & 2(c_2a_2 - c_3a_1) & 0 & b_2c_2 & -b_2c_3 \\ & & 0 & 2(a_3c_3 - c_1a_2) & -b_3a_2 & b_3a_3 & -b_3c_1 & 0 & b_3c_3 \\ & & & 0 & \begin{matrix} \frac{a_2a_1}{2} + \\ c_1b_2 - c_3c_2 \end{matrix} & \begin{matrix} -\frac{a_2a_3}{2} + \\ c_1c_2 - b_1c_3 \end{matrix} & \frac{a_1c_1}{2} & 0 & -\frac{a_1c_3}{2} \\ & & & & 0 & \begin{matrix} \frac{a_2a_3}{2} + \\ b_3c_2 - c_1c_3 \end{matrix} & -\frac{a_2c_1}{2} & \frac{a_2c_2}{2} & 0 \\ & & * & & & 0 & 0 & -\frac{a_3c_2}{2} & \frac{a_3c_3}{2} \\ & & & & & & 0 & -\frac{c_1c_2}{2} & \frac{c_1c_3}{2} \\ & & & & & & & 0 & -\frac{c_2c_3}{2} \\ & & & & & & & & 0 \end{pmatrix}$$

$$P_{\mathrm{Lin}} = \begin{pmatrix} 0 & 0 & 0 & -2c_2 & 0 & 2c_1 & -2a_3 & 2a_1 & 0 \\ & 0 & 0 & c_2 & -2c_3 & 0 & 0 & -2a_1 & 2a_2 \\ & & 0 & 0 & 2c_3 & -2c_1 & 2a_3 & 0 & -2a_2 \\ & & & 0 & -a_3 & a_2 & -c_3 & b_2-b_1 & c_1 \\ & & & & 0 & -a_1 & c_2 & -c_1 & b_3-b_2 \\ & & & & & 0 & b_1-b_3 & c_3 & -c_2 \\ & & & * & & & 0 & a_2 & -a_1 \\ & & & & & & & 0 & a_3 \\ & & & & & & & & 0 \end{pmatrix}.$$

The Hamiltonians satisfying Theorem 11 in this case are:

$$H_{\mathrm{Skl}} = H_{13} = b_1 + b_2 + b_3,$$
$$H_{\mathrm{Lin}} = H_{22} = a_1c_2 + a_2c_3 + a_3c_1,$$
$$H_{\mathrm{Sem}} = H_{12} = a_1^2 + a_2^2 + a_3^2 - b_1b_2 - b_1b_3 - b_2b_3 + c_1^2 + c_2^2 + c_3^2.$$

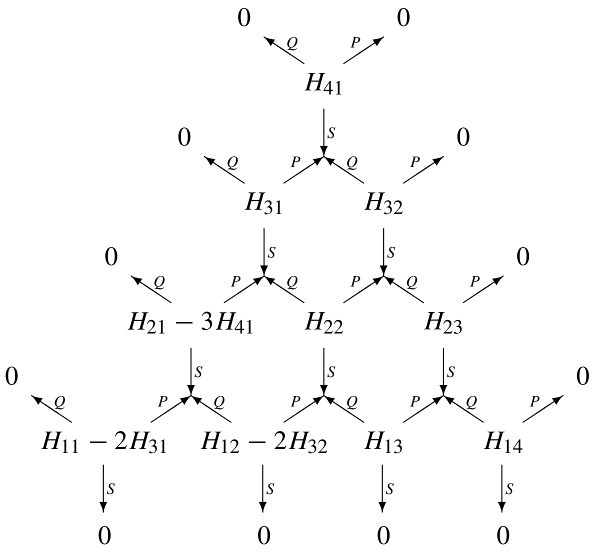
Moreover, using all the three Poisson structures, it is possible to construct the following recursion scheme, that prove the complete integrability of the system:

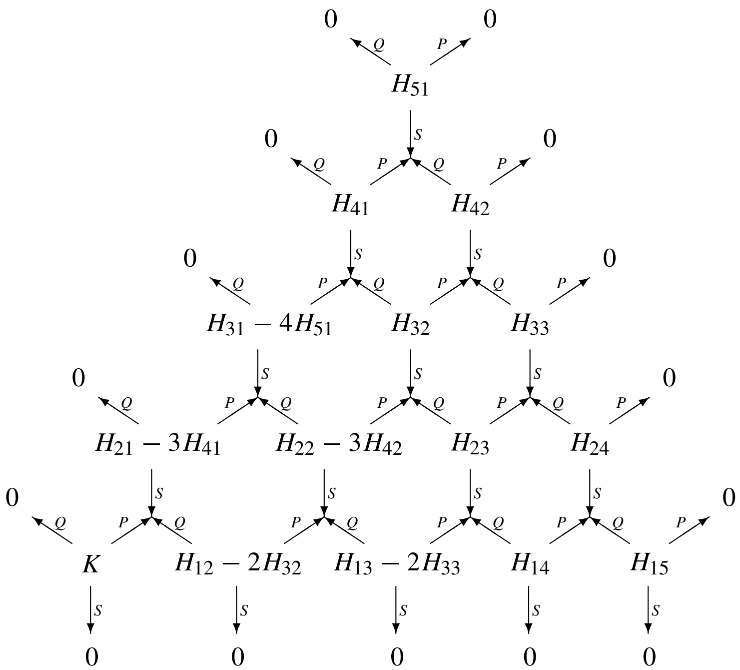


where $P = P_{\mathrm{Sem}}$, $Q = P_{\mathrm{Skl}}$, $S = P_{\mathrm{Lin}}$ and

$$H_{11} = b_1b_2b_3 - b_1(a_2^2 + c_3^2) - b_2(a_3^2 + c_1^2) - b_3(a_1^2 + c_2^2) - 2a_1a_2c_1 - 2a_2a_3c_2 - 2a_3a_1c_3 - 2c_1c_2c_3,$$
$$H_{21} = a_1a_2a_3 + a_1(c_1c_3 - c_2b_3) + a_2(c_2c_1 - c_3b_1) + a_3(c_3c_2 - c_1b_2),$$
$$H_{31} = c_1c_2c_3.$$

Analogous results can be obtained also for higher dimensions: in the cases $n = 4$ and $n = 5$, by direct computation, one can respectively construct the recursion schemes





where

$$\det(L_\lambda - \mu) = (-\mu)^n + H_1(\lambda + \lambda^{-1}) + \cdots + H_n(\lambda^n + \lambda - n)$$

$$H_k = \sum_{j=1}^{n-k+1} H_{kj}\mu^j, \qquad K = H_{11} - 2H_{31} + 2H_{51}$$

for $L_\lambda \in \mathfrak{G}^+_{n-1} \subset \mathfrak{gl}(n)[\lambda, \lambda^{-1}]$.

The regularity in the schemes leads one naturally to suppose that the construction is possible for all n, i.e. to conjecture that the extension of the periodic Toda lattice with n particles introduced in this paper is not only trihamiltonian, but also completely integrable.

5. Conclusions

In this paper a trihamiltonian structure is constructed on a subspace of the Kupershmidt algebra of shift operators on n-periodic sequences. This subspace is formed by symmetric operators of maximum degree $n-1$ and it is isomorphic to the space $\mathfrak{H}$ of $n \times n$ Hermitian matrices. The first two compatible Poisson brackets that define the trihamiltonian structure are the well known Semënov–Tian–Shansky and Sklyanin brackets obtained from r-matrix techniques. The third Poisson bracket is the Lie–Poisson structure on $\mathfrak{H}$. As an application a trihamiltonian extension of the periodic Toda lattice with n particles is constructed, and a complete set of first integrals mutually in involution for the extended system up to five particles is found using a trihamiltonian recursion scheme.

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Appendix

In this appendix the expression (20) appearing in Proposition 8 is proved. If $\tilde{f} = f \circ \Phi$ and $M = \Phi(M_\lambda)$, one can check the identity

$$(\nabla \tilde{f}, M_\lambda) = (\!(\nabla f, M)\!) \quad \forall M_\lambda \in \mathfrak{G}^+_{n-1}.$$

Thus, the definitions $F = \nabla f$ and $F_\lambda = \nabla \tilde{f}$ are consistent with the notation $F = \Phi(F_\lambda)$.

Moreover, if $M_\lambda \in \mathfrak{G}_+$ (i.e. $*M_\lambda = -M_\lambda$ holds), then

$$M_\lambda = M_0 + \sum_{k>0} M_k \lambda^k - \sum_{k>0} M_k^t \lambda^{-k}$$
$$(M_\lambda)^+ = \rho(M_0) + \sum_{k>0} M_k \lambda^k + \sum_{k>0} M_k^t \lambda^{-k}$$

while if $M_\lambda \in \mathfrak{G}^+$ (i.e. $*M_\lambda = M_\lambda$ holds) then

$$M_\lambda = M_0 + \sum_{k>0} M_k \lambda^k + \sum_{k>0} M_k^t \lambda^{-k}$$
$$(M_\lambda)_+ = \rho(M_0) + \sum_{k>0} M_k \lambda^k - \sum_{k>0} M_k^t \lambda^{-k}.$$

These two facts follow immediately from the definitions of the two projections on $\mathfrak{G}_+$ and $\mathfrak{G}^+$.

Using the representation (14) for a Hamiltonian vector field for the Sklyanin bracket one has

$$\begin{aligned}\{\tilde{f}, \tilde{g}\}_{\mathrm{Skl}} = (G_\lambda, X^{\mathrm{Skl}}_{\tilde{f}}) &= (\!(G, \Phi([L_\lambda, (L_\lambda \bullet F_\lambda)_+] - L_\lambda \bullet [L_\lambda, F_\lambda]^+))\!) \\ &= (\!(G, [\![i\, \Phi((L_\lambda \bullet F_\lambda)_+), L]\!] - L \odot \Phi([L_\lambda, F_\lambda]^+))\!)\end{aligned}$$

and therefore it is essential to calculate $\Phi\,((L_\lambda \bullet F_\lambda)_+)$ and $\Phi\left([L_\lambda, F_\lambda]^+\right)$ for $L_\lambda, F_\lambda \in \mathfrak{G}^+_{n-1}$. Using the identities

$$L_1 \bullet F_1^t + L_1^t \bullet F_1 = L_1 \bullet F_1 + L_1^t \bullet F_1^t - L_A \bullet F_A$$
$$[L_1, F_1^t] + [L_1^t, F_1] = [L_1, F_1] + [L_1^t, F_1^t] - [L_A, F_A]$$
$$\rho(L_1 \bullet F_1) = -L_1 \bullet F_1, \qquad \rho(L_1^t \bullet F_1^t) = L_1^t \bullet F_1^t$$
$$\rho([L_1, F_1]) = -[L_1, F_1], \qquad \rho([L_1^t, F_1^t]) = [L_1^t, F_1^t]$$

a tedious but straightforward computation gives

$$\begin{aligned}[L_\lambda, F_\lambda]^+ &= \rho([L_S, F_S] - [L_A, F_A]) - [L_1, F_1] + [L_1^t, F_1^t] + \lambda^2[L_1, F_1] - \lambda^{-2}[L_1^t, F_1^t] \\ &\quad + \lambda([L_S, F_1] + [L_1, F_S]) - \lambda^{-1}([L_S, F_1^t] + [L_1^t, F_S]) \\ (L_\lambda \bullet F_\lambda)_+ &= \rho(L_S \bullet F_S - L_A \bullet F_A) - L_1 \bullet F_1 + L_1^t \bullet F_1^t + \lambda^2 L_1 \bullet F_1 - \lambda^{-2} L_1^t \bullet F_1^t \\ &\quad + \lambda(L_S \bullet F_1 + L_1 \bullet F_S) - \lambda^{-1}(L_S \bullet F_1^t + L_1^t \bullet F_S).\end{aligned}$$

Finally, applying Φ and using the identities

$$\begin{aligned}&L_1 + L_1^t = -\mathtt{s}(L), \qquad L \bullet \mathtt{s}(M) = L \odot \mathtt{s}(M), \qquad i[L, \mathtt{s}(M)] = [\![L, \mathtt{s}(M)]\!], \\ &i\rho(L_S \bullet F_S - L_A \bullet F_A) = \mathtt{a}(L \odot F), \qquad \rho([L_S, F_S] - [L_A, F_A]) = \mathtt{s}([\![L, F]\!])\end{aligned}$$

and

$$\begin{aligned}&2L_1^t \bullet F_1^t - 2L_1 \bullet F_1 = \mathtt{s}(L) \odot F_A + L_A \odot \mathtt{s}(F) \\ &2[L_1, F_1] + 2[L_1^t, F_1^t] = i[\![\mathtt{s}(L), F_A]\!] + i[\![L_A, \mathtt{s}(F)]\!]\end{aligned}$$

one obtains

$$\begin{aligned}i\Phi((L_\lambda \bullet F_\lambda)_+) &= i\rho(L_S \bullet F_S - L_A \bullet F_A) + i\mathtt{s}(L) \odot F_A + iL_A \odot \mathtt{s}(F) + L_S \odot \mathtt{s}(F) + \mathtt{s}(L) \odot F_S \\ &= \mathtt{a}(L \odot F) + L \odot \mathtt{s}(F) + \mathtt{s}(L) \odot F \\ -\Phi\left([L_\lambda, F_\lambda]^+\right) &= -\rho([L_S, F_S] - [L_A, F_A]) + i[\![\mathtt{s}(L), F_A]\!] + i[\![L_A, \mathtt{s}(F)]\!] + [\![L_S, \mathtt{s}(F)]\!] + [\![\mathtt{s}(L), F_S]\!] \\ &= -\mathtt{s}([\![L, F]\!]) + [\![L, \mathtt{s}(F)]\!] + [\![\mathtt{s}(L), F]\!].\end{aligned}$$

Hence, the expression (20) in Proposition 8 is proved:

$$\begin{aligned}&(\!(G, [\![i\Phi((L_\lambda \bullet F_\lambda)_+), L]\!] - L \odot \Phi([L_\lambda, F_\lambda]^+))\!) \\ &\quad = (\!(G, [\![\mathtt{a}(L \odot F) + \mathtt{s}(L) \odot F, L]\!] - L \odot \mathtt{s}([\![L, F]\!]) - \mathtt{s}(L) \odot [\![L, F]\!] + [\![L \odot \mathtt{s}(L), F]\!])\!) \\ &\quad = (\!(L \odot \mathtt{s}(L), [\![F, G]\!])\!) - (\!(L, [\![F \odot \mathtt{s}(L) + \mathtt{a}(L \odot F), G]\!] + [\![F, G \odot \mathtt{s}(L) + \mathtt{a}(L \odot G)]\!])\!).\end{aligned}$$

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